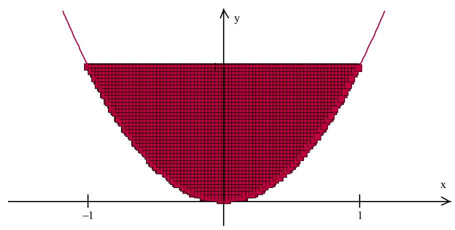
Archimedes and Squaring the Parabola

Minor Source: The Archimedes Codex, Reviel Netz and William Noel, pages 45-52 (2007). Actually this is the book that suggested the idea. I wasn't happy with their approach (meaning I had trouble following it!) so I did my own. The book is about the discovery of an unknown and hugely important book of Archimedes that was largely "erased" and covered over in the 12th century by a recycler monk.

Suggestion: Don't read this without graph paper and pencil handy!

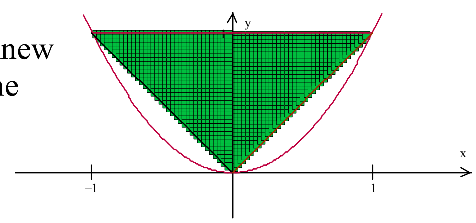
The term 'squaring the ___' meant to find the area. The classical Greeks didn't have the number system or algebraic emphasis that we do, so they found the area of an object by finding the length of a side of a square that had the same area as the object in question.

But what does it mean to find the area of a parabola?

Let me show you what the idea was: 

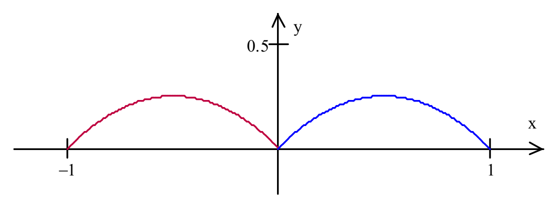
How do we find the shaded area? As you know from Geometry, the Greeks were experts at polygons (straight line edges). And they showed us so much about circles. But what about other shapes with curvy edges?

Archimedes' idea was to create a triangle inside the region – after all, he knew how to find the area of a triangle. So, here's our picture … I've chosen the vertices to be (-1,1), (0,0), and (1,1) to make my life easy….



So obviously the area of our original region is larger than the area of the triangle which is $A_1 = \frac{1}{2}(2)(1) = 1$.

But what about the area of those unshaded pieces? Here's the idea that should make the method completely accessible to you …. That seems like a difficult problem. What we're going to do is subtract the parabola from the bottom of the triangle. So we have $y = -x - x^2$ on the left, and $y = x - x^2$ on the right. Another graph should help ….



The regions are not the same shape. Even if you take into account the scales which don't agree …. But the areas of the regions are equal! This is Cavalieri's principle. If each and every corresponding cross-sections (along one direction) are identical, the areas are the same. It applies to volumes as well. Think of a stack of quarters. The volume of the coins is the same regardless of how 'tilted' your stack is. Or if you've ever tried to make a seven-layer cake and transport it – and the layers slide to one side – there still is the same amount of cake. (Brother's wedding; worst snowstorm in Yonkers' history … long story).

Focus on the region on the right (between x=0 and x=1). Approximate *that* area with one triangle, and find the area of that triangle (call it A_2).

So our approximation of the area of the original region is $A_1 + 2A_2$ (remember we have two of the smaller regions).

Graph the result … and repeat the process (subtract, pick one smaller region, approximate, ….) You will get a series (remember the first month of PreCalc?)

This, again, is the method of exhaustion. We don't calculate the exact area of our original region – or at least Archimedes couldn't – you can because you know how to sum geometric series) but you can get as close as you need to! It's actually an amazing result considering this was 2300 years ago, before algebra!

Archimedes' approach actually constructed a proof of the relationship he found – one of the pieces of evidence that clearly marks him as the greatest mathematician before 1600 AD.

Extra Credit:

You can follow the same process that Archimedes did to get a series of approximations to the original area. Try to identify a formula that gives you each term. Use the common ratio to calculate the *infinite* sum, yielding a calculation of the relationship that Archimedes found. Take your place alongside the greatest mathematician of antiquity, and if you write it up real nicely for me take all 3 points available

Other sources:

I haven't taken the time to look … but do let me know if you find any good ones!